Advanced Probability : Back-Paper Exam

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Submit solutions via Moodle by 15th December 12:30 PM.

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Please write and sign the following declaration on your answer script first :

I have not received, I have not given, nor will I give or receive, any assistance to another student taking this exam, including discussing the exam with other students. The solution to the problems are my own and I have not copied it from anywhere else. I have used only class notes and the notes of D. Panchenko, R. Durrett and M. Krishnapur.

Attempt any five questions. Each question carries 10 points. If you attempt more than five questions, the first five answers will be evaluated.

1. Let $X_i, i \geq 1$ be i.i.d. random variables such that $\mathbb{P}(X_1 = +1) = p = 1 - \mathbb{P}(X_1 = -1)$. Consider the random walk $S_n := \sum_{i=1}^n X_i$. Let $p > \frac{1}{2}$ and $q = 1-p$. Consider the integer $b \geq 1$ and let $\tau := \min\{n \geq 1 : S_n = b\}$. Show that for $0 < s \leq 1$,

$$\mathbb{E}[s^\tau] = (\frac{1 - (1 - 4pqs^2)^{\frac{1}{2}}}{2qs})^b,$$

and compute $\mathbb{E}[\tau]$.

2. Let M_n be a Poisson(n) random variable. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. uniform random vectors in the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$ and are also independent of M_n. Let $B_1, \ldots, B_k$ be Borel subsets of D. Let

$$N_{i,n} := |\{X_1, \ldots, X_{M_n}\} \cap B_i|, 1 \leq i \leq k$$

be the number of points $X_1, \ldots, X_n$ falling inside B_i. Consider the vector $N_n = (N_{1,n}, \ldots, N_{k,n}), n \geq 1$. Is there a vector μ_n and scalar $\sigma_n \geq 0$ such that

$$\frac{N_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0, C),$$

for some matrix C ? If yes, find μ_n, σ_n and C as well.

3. Let $Z_n, n \geq 0$ be the Galton-Watson branching process with mean off-spring μ i.e. $X_{i,j}, i, j \geq 1$ are i.i.d. $\mathbb{Z}_+$-valued random variables with pmf $(p_k)_{k \geq 0}$ and
$$Z_0 := 1, Z_{n+1} := \sum_{j=1}^{Z_n} X_{n+1,j}, n \geq 0.$$
Let $\mu = \mathbb{E}[X_{1,1}]$. Define $\phi(s) := \sum_{k=0}^{\infty} s^k p_k, s \in [0,1]$ as the probability generating function of $X_{1,1}$ and let s_0 be the smallest root of $\phi(s) = s$ in $s \in [0,1]$. Show that Z_n/μ^n and $(s_0)^{Z_n}$ are martingales.

4. Let $X = (X_1, \dots, X_k)$ be a multivariate Normal random variable on $\mathbb{R}^k$ with distribution $N(0, C)$. Prove that
$$\mathbb{E}[X_1 F(X)] = \sum_{i=1}^{n} C_{1i} \mathbb{E}[\frac{\partial F}{\partial x_i}(X)],$$
for $F : \mathbb{R}^k \to \mathbb{R}$ with second partial derivatives and integrable first partial derivatives.

5. Suppose that X_i are i.i.d. random variables such that $\mathbb{E}[X_1] = 0, \mathbb{E}[|X_1|] < \infty$. If $c_n, n \geq 1$ is a bounded sequence of real numbers, show that as $n \to \infty$,
$$\frac{1}{n} \sum_{i=1}^{n} c_i X_i \overset{a.s.}{\to} 0.$$

6. Let $f : [0,1]^k \to \mathbb{R}$ be a continuous function. Show that
$$\lim_{n\to\infty} \sum_{0 \leq j_1, \dots, j_k \leq n} f(\frac{j_1}{n}, \dots, \frac{j_k}{n}) \prod_{i=1}^{k} \binom{n}{j_i} x_i^{j_i} (1 - x_i)^{n-j_i} = f(x_1, \dots, x_k),$$
, uniformly on $[0,1]^k$.